Non-Negative Matrices: The Open Leontief Model

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ABSTRACT

The consequences of arbitrary changes of the final demand vector for the gross production vector in the open Leontief model are studied, and several properties of non-negative irreducible square matrices are obtained.

1. PRELIMINARY REMARKS AND DEFINITIONS

In this paper T will invariably be used for a non-negative irreducible (n,n) matrix with $n\in\mathbb{N}$; $\mathbb{N}$ is the set of natural numbers $\geqslant 1$. For each $n\in\mathbb{N}$ we shall write $\langle n\rangle=\{1,\dots,n\}$. The matrix $T=\{t_{ij}\}$ is called *non-negative* if and only if $t_{ij}\geqslant 0$ for each $i,j\in\langle n\rangle$, and is called *irreducible* if and only if for each $i,j\in\langle n\rangle$ there is an element $k\in\mathbb{N}$, dependent on i and j, such that $t_{ij}^{(k)}>0$ with $T^k=\{t_{ij}^{(k)}\}$; see e.g. [5], Definition 1.6. We then define $T^0=I$, the (n,n) identity matrix. The ith unit vector of $\mathbb{R}^n$ is denoted by $\mathbf{e}_i=(0,\dots,0,1,0,\dots,0)'$ for each $i\in\langle n\rangle$, and we shall write $\mathbf{1}=(1,\dots,1)'\in\mathbb{R}^n$.

For any $\mathbf{x}\in\mathbb{R}^n$ we shall use the following notations:

$$\begin{aligned}
&\mathbf{x}\geqslant\mathbf{0} \quad \text{if } x_i\geqslant 0 \text{ for each } i\in\langle n\rangle,\\
&\mathbf{x}>\mathbf{0} \quad \text{if } x\geqslant 0 \text{ and } \mathbf{x}\neq\mathbf{0},\\
&\mathbf{x}\gg\mathbf{0} \quad \text{if } x_i>0 \text{ for each } i\in\langle n\rangle.
\end{aligned}$$

Similar notations are used for matrices.

The symbol s will denote any positive real number, and r the *Perron-Frobenius eigenvalue* of T; see e.g. [5], Theorem 1.5. It is clear that $T\mathbf{1}$ is the vector of row sums of T and $T'\mathbf{1}$ the vector of column sums of T. Hence, $(sI-T)\mathbf{1}>\mathbf{0}$ means that all row sums of T are less than or equal to s and at least one row sum is less than s; $(sI-T')\mathbf{1}>\mathbf{0}$ means the same for the

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column sums of T. In [2] the so-called M(inkowski)-matrices are studied. An M-*matrix* is any matrix of the form $sI - A$, with $s > 0$, $A \geqslant 0$, and $r \geqslant s$, where r is the Perron-Frobenius root of A. Many properties of M-matrices are given in [2, Chapter 6], and an extensive bibliography can be found in this book.

2. THE MINKOWSKI THEOREM

The following proposition is the well-known Minkowski theorem; see e.g. [6], Theorem 4.C.10.

PROPOSITION 1. $(sI - T)\mathbf{1} > \mathbf{0}$ *or* $(sI - T')\mathbf{1} > \mathbf{0} \Rightarrow r < s$.

Proof. The following inequalities for r are direct consequences of the Perron-Frobenius theorem (see e.g. [5], Theorem 1.5):

$$\min_{i \in \langle n \rangle} \sum_{j=1}^{n} t_{ij} \leqslant r \leqslant \max_{i \in \langle n \rangle} \sum_{j=1}^{n} t_{ij},$$

with equality on either side implying equality throughout. Similar inequalities hold for the column sums of T. If all row sums of T are equal to α, then $r = \alpha$ and it follows from $(s - \alpha)\mathbf{1} > \mathbf{0}$ that $r = \alpha < s$. If not all row sums are equal, then $r < \max_{i \in \langle n \rangle} \sum_{j=1}^{n} t_{ij}$, and it also follows that $r < s$. ∎

The converse of the above proposition is not true in general, for consider the matrix

$$T = \begin{bmatrix} \frac{1}{22} & \frac{15}{22} \\ \frac{15}{22} & \frac{8}{22} \end{bmatrix}.$$

It is obvious that $r = \frac{1}{44}(9 + \sqrt{949}\,) < \frac{40}{44} < 1$. Taking $s = 1$, we find that $r < s$. However, there is both a row sum and a column sum which exceeds unity.

3. THE OPEN LEONTIEF MODEL

We shall consider the following equation system:

$$(sI - T)\mathbf{x} = \mathbf{c} \tag{1}$$

with $\mathbf{c} > \mathbf{0}$ and $\mathbf{x} \in \mathbb{R}^n$.

For $s = 1$ the equation system (1) can be interpreted as an *open Leontief (input-output) model*, with $\mathbf{c}$ being the *(final) demand vector* and $\mathbf{x}$ the *(gross) production vector*; T is then the input-coefficient matrix of the model. An introductory discussion of the open Leontief model can be found in R.G.D. Allen [1, Chapter 11].

It is common to consider also the "dual system":

$$(sI - T')\mathbf{p} = \mathbf{v}, \tag{2}$$

T' being the transpose of T, $\mathbf{v} > \mathbf{0}$, and $p \in \mathbb{R}^n$. Then $\mathbf{v}$ is interpreted as the vector of the *value added* of the commodities, and $\mathbf{p}$ the vector of *prices* of the commodities. A link between the models (1) and (2) is given by the relation

$$\sum_{j=1}^{n} v_j x_j = \sum_{i=1}^{n} c_i p_i, \tag{3}$$

interpreting $\mathbf{x} = (x_1, \ldots, x_n)'$, $\mathbf{c} = (c_1, \ldots, c_n)'$, $\mathbf{p} = (p_1, \ldots, p_n)'$, and $\mathbf{v} = (v_1, \ldots, v_n)'$. The expression (3) can be interpreted as follows: the "national income" and the "national product" are equal.

4. THE INTERPRETATION OF ROW AND COLUMN SUMS

The expression $(sI - T)\mathbf{1} > \mathbf{0}$—all row sums of T are less than or equal to s, and at least one row sum is less than s—can be interpreted for $s = 1$ as follows: suppose that of each commodity exactly one unit is produced. Then $\sum_{j=1}^{n} t_{ij}$ is the total amount of commodity i required for the production of this particular combination of commodities. It is clear that this amount may not exceed the production of commodity i, and that at least one commodity is available for demand outside the industry section, i.e., the final demand.

The expression $(sI - T')\mathbf{1} > \mathbf{0}$—all column sums of T are less than or equal to s, and at least one column sum is less than s—can be interpreted in the following way: choose the measuring unit of each commodity to be such that the price of each commodity equals one (e.g. in dollars). Then $\sum_{i=1}^{n} t_{ij}$ is the total amount the jth industry pays (per unit) for raw materials and intermediate commodities. The ith industry receives one (dollar) per unit. Hence, the expression above means that the value added (output per unit) is non-negative for all commodities and strictly positive for at least one commodity. So at least one industry is able to pay the factors labor and capital. See also [6], p. 364.

In [5, p. 31] Seneta writes that "it is difficult to ascribe a direct meaning in economic terms to the condition $\sum_{j=1}^{n} t_{ij} \leqslant 1$ with strict inequality for at least one i." His interpretation of "all column sums are less than or equal to unity and at least one is less than unity"—viz.: if commodities are measured in dollar worth units, no industry operates at a loss and at least one operates at a profit in terms of internal ("nonlabor") factors—is not based on the usual assumptions of the open Leontief model (see [1], Chapter 11). An interpretation is given above.

5. CHANGING THE DEMAND VECTOR

The following proposition plays an essential part throughout this paper.

PROPOSITION 2. *The equation system* (1) *has a solution* $\mathbf{x}>\mathbf{0}$ *for each* $\mathbf{c}>\mathbf{0}$ *if and only if* $r<s$. *In this case there is only one solution* $\mathbf{x}\gg\mathbf{0}$, *which is given by*

$$\mathbf{x} = (sI - T)^{-1}\mathbf{c}. \tag{4}$$

Proof. See [5], Theorem 2.1, and also [2], Theorem 2.3 (N38 and N39). ■

For each $\mathbf{c}_1, \mathbf{c}_2 > 0$ define $\Delta\mathbf{c} \in \mathbb{R}^n$ by

$$\Delta\mathbf{c} = \mathbf{c}_2 - \mathbf{c}_1. \tag{5}$$

In virtue of Proposition 2 there are vectors $\mathbf{x}_1, \mathbf{x}_2 \gg \mathbf{0}$ such that $(sI - T)\mathbf{x}_1 = \mathbf{c}_1$ and $(sI - T)\mathbf{x}_2 = \mathbf{c}_2$, provided $r<s$. Define $\Delta\mathbf{x} \in \mathbb{R}^n$ by

$$\Delta\mathbf{x} = \mathbf{x}_2 - \mathbf{x}_1. \tag{6}$$

PROPOSITION 3. *Let* $r<s$. *Then the following assertions hold:*

(a) $\Delta\mathbf{c} \neq \mathbf{0} \Leftrightarrow \Delta\mathbf{x} \neq 0$;
(b) $\Delta\mathbf{c} > \mathbf{0} \Rightarrow \Delta\mathbf{x} \gg \mathbf{0}$;
(c) $\Delta\mathbf{c} < \mathbf{0} \Rightarrow \Delta\mathbf{x} \ll \mathbf{0}$.

Proof.

(a) Note that $\Delta\mathbf{c} = \mathbf{c}_2 - \mathbf{c}_1 = (sI - T)(\mathbf{x}_2 - \mathbf{x}_1) = (sI - T)\Delta\mathbf{x}$. Hence,

$$(sI - T)\Delta\mathbf{x} = \Delta\mathbf{c}. \tag{7}$$

If $\Delta\mathbf{x}=\mathbf{0}$, then (7) implies that $\Delta\mathbf{c}=\mathbf{0}$. So $\Delta\mathbf{c}\neq\mathbf{0}$ implies that $\Delta\mathbf{x}\neq 0$. As $r<s$, Proposition 2 implies that $(sI-T)^{-1}\gg 0$. Therefore,

$$(sI-T)^{-1}\Delta\mathbf{c} = \Delta\mathbf{x}. \tag{8}$$

If $\Delta\mathbf{c}=\mathbf{0}$, it follows from (8) that $\Delta\mathbf{x}=\mathbf{0}$. Hence, $\Delta\mathbf{x}\neq\mathbf{0}$ implies $\Delta\mathbf{c}\neq\mathbf{0}$.

(b) Take any $\Delta\mathbf{c}>\mathbf{0}$. Proposition 2 implies directly that $\Delta\mathbf{x}\gg\mathbf{0}$.

(c) As $\Delta\mathbf{c}\ll\mathbf{0}$ implies $-\Delta\mathbf{c}\gg\mathbf{0}$, it follows from (7) that $(sI-T)(-\Delta\mathbf{x})=(-\Delta\mathbf{c})$. Again Proposition 2 implies that $-\Delta\mathbf{x}\gg\mathbf{0}$; hence $\Delta\mathbf{x}\ll\mathbf{0}$. ∎

Taking $s=1$ in the above theorem, we obtain, in terms of the open Leontief model, the following interpretations:

(a) means that the production of at least one commodity changes if and only if the demand for at least one commodity changes.

(b) means that if the demand for at least one commodity increases, then the production of all commodities increases.

(c) means the same as (b) for a decrease of the demand.

However, it may so happen that $\Delta x_j=0$ and $\Delta c_j>0$ for some $j\in\langle n\rangle$, as the following example shows:
Take

$$T=\frac{1}{4}\begin{bmatrix}2&1&1\\1&1&1\\1&1&2\end{bmatrix}.$$

Then

$$(I-T)^{-1}=\frac{4}{3}\begin{bmatrix}5&3&4\\3&3&3\\4&3&5\end{bmatrix}.$$

Taking $\Delta\mathbf{c}=(-1,1,0)'$, it follows that $\Delta\mathbf{x}=\frac{4}{3}(-2,0,-1)'$. Hence, $\Delta c_2=1$ and $\Delta x_2=0$.

The converse of Theorem 3(b) is not true in general. Take for instance $\Delta\mathbf{c}=(2,0,-1)'$. Then $\Delta\mathbf{x}=\frac{4}{3}(6,3,3)'\gg\mathbf{0}$. Similarly, the converse of (c) is not true in general.

Note that Theorem 3 is a generalization of one due to M. Morishima [4, Chapter I, Theorem 6 (first part)].

Replacing T by its transpose T', $\mathbf{c}$ by $\mathbf{v}$, and $\mathbf{x}$ by $\mathbf{p}$, we obtain the equation system (2). Proposition 2 now implies that (2) has for each $\mathbf{v}>\mathbf{0}$ a solution $\mathbf{p}>\mathbf{0}$ if and only if $s>r$; in this case there is only one solution $\mathbf{p}\gg\mathbf{0}$, which is given by $\mathbf{p}=(sI-T')^{-1}\mathbf{v}$. Theorem 3 also implies that:

(a′) $\Delta\mathbf{v}\neq\mathbf{0} \Leftrightarrow \Delta\mathbf{p}\neq\mathbf{0}$;
(b′) $\Delta\mathbf{v}>\mathbf{0} \Rightarrow \Delta\mathbf{p}\gg\mathbf{0}$;
(c′) $\Delta\mathbf{v}<\mathbf{0} \Rightarrow \Delta\mathbf{p}\ll\mathbf{0}$;

provided $r<s$. The interpretation of (b′) for instance is: if the value added of at least one commodity increases, then the prices of all commodities increase.

Throughout this paper we shall give the interpretation in terms of the model (1) only; interpretations in terms of (2) are left to the reader. We also note that, generally speaking, only the assertions of the theorems are interpreted; the conditions under which the assertions hold are mostly omitted in the interpretations.

6. ON POWERS OF THE MATRIX T

The next two lemmas are subsidiary to the proof of a theorem concerning absolute changes of the production vector in the system (1).

LEMMA 4. *Let $k\in\mathbb{N}$. Then the following assertions hold:*

(a) $(sI-T)\mathbf{1}\geqslant\mathbf{0} \Leftrightarrow (s^kI-T^k)\mathbf{1}\geqslant\mathbf{0}$;
(b) $(sI-T)\mathbf{1}\gg\mathbf{0} \Leftrightarrow (s^kI-T^k)\mathbf{1}\gg\mathbf{0}$;
(c) $(sI-T)\mathbf{1}>\mathbf{0} \Leftrightarrow (s^kI-T^k)\mathbf{1}>\mathbf{0}$, *provided* $T\gg 0$ *and* $k\geqslant 2$.

Proof.

(a) If $(sI-T)\mathbf{1}\geqslant\mathbf{0}$, then $T\mathbf{1}\leqslant s\mathbf{1}$, and hence $T^2\mathbf{1}=T(T\mathbf{1})\leqslant T(s\mathbf{1})\leqslant s^2\mathbf{1}$. By induction on k it follows that $T^k\mathbf{1}\leqslant s^k\mathbf{1}$, and therefore, that $(s^kI-T^k)\mathbf{1}\geqslant\mathbf{0}$. The reverse of this implication is trivial.

(b) The proof is similar to the proof of (a): replace $\geqslant$ by $\gg$.

(c) Let $(sI-T)\mathbf{1}>\mathbf{0}$; we may then assume that $\sum_{j=1}^n t_{ij}<s$ for certain $i\in\langle n\rangle$. The proof is by induction on k. First take $k=2$. As $\sum_{j=1}^n t_{ij}^{(2)}=\sum_{k=1}^n t_{ik}\sum_{m=1}^n t_{km}$, $t_{ik}>0$ for each $k\in\langle n\rangle$, and $\sum_{m=1}^n t_{km}<s$ for some $k\in\langle n\rangle$, it follows that $\sum_{j=1}^n t_{ij}^{(2)}<s\sum_{k=1}^n t_{ik}\leqslant s^2$ for each $i\in\langle n\rangle$. Now assume that $(s^{k-1}I-T^{k-1})\mathbf{1}\gg\mathbf{0}$, and hence $\sum_{j=1}^n t_{ij}^{(k-1)}<s^{k-1}$ for each $j\in\langle n\rangle$. Then

$$\sum_{j=1}^n t_{ij}^{(k)}=\sum_{h=1}^n\sum_{j=1}^n t_{ij}t_{jh}^{(k-1)}=\sum_{j=1}^n t_{ij}\sum_{h=1}^n t_{jh}^{(k-1)}$$

$$<\sum_{j=1}^n t_{ij}s^{k-1}\qquad(\text{because } t_{ij}>0 \text{ for each } i,j\in\langle n\rangle)$$

$$< s\cdot s^{k-1} = s^k \qquad \text{for each} \quad i\in\langle n\rangle.$$

Therefore we have in fact that $(s^kI-T^k)\mathbf{1}\gg\mathbf{0}$. ∎

Note that the irreducibility of T is not used in the above proof, so Lemma 4 holds for any non-negative (n,n) matrix T.

LEMMA 5. *For each $k\in\mathbb{N}$ the following holds:*

$$(s^kI-T^k)\mathbf{x} = \sum_{m=1}^{k} s^{k-m}T^{m-1}\mathbf{c} \tag{9}$$

Proof.[1] The proof of (9) is by induction on k. For $k=1$ (9) is undoubtedly true, because in this case (9) is equivalent to (1). Take $k\geqslant 2$ and assume that (9) holds for $k-1$. We will show that (9) also holds for k:

$$\begin{aligned} T^k\mathbf{x} &= T(T^{k-1}\mathbf{x}) = T\left(s^{k-1}\mathbf{x}-\sum_{m=1}^{k-1} s^{k-m-1}T^{m-1}\mathbf{c}\right) \\ &= s^{k-1}(T\mathbf{x})-\sum_{m=1}^{k-1} s^{k-m-1}T^{m}\mathbf{c} = s^{k-1}(s\mathbf{x}-\mathbf{c})-\sum_{m=1}^{k-1} s^{k-m-1}T^{m}\mathbf{c} \\ &= s^k\mathbf{x}-s^{k-1}T^o\mathbf{c}-\sum_{m=2}^{k} s^{k-m}T^{m-1}\mathbf{c} = s^k\mathbf{x}-\sum_{m=1}^{k} s^{k-m}T^{m-1}\mathbf{c}. \end{aligned}$$

Hence, $(s^kI-T^k)\mathbf{x}=\sum_{m=1}^{k}s^{k-m}T^{m-1}\mathbf{c}$. ∎

7. POSITIVITY AND NEGATIVITY OF THE CHANGE OF THE PRODUCTION VECTOR

For each $\mathbf{z}\in\mathbb{R}^n$ we define

$$\begin{aligned} \Gamma(\mathbf{z}) &= \{i\,|\,i\in\langle n\rangle \text{ and } z_i\neq 0\}, \\ \Gamma_+(\mathbf{z}) &= \{i\,|\,i\in\langle n\rangle \text{ and } z_i> 0\}, \\ \Gamma_-(\mathbf{z}) &= \{i\,|\,i\in\langle n\rangle \text{ and } z_i< 0\}. \end{aligned}$$

[1]The lemma also follows from $\sum_{m=1}^{k}s^{k-m}T^{m-1}=(s^kI-T^k)(sI-T)^{-1}$.

Note that $\Gamma(\mathbf{z})=\Gamma_+(\mathbf{z})\cup\Gamma_-(\mathbf{z})$, and that $\Gamma_+(\mathbf{z})\cap\Gamma_-(\mathbf{z})=\varnothing$. Also, note that $\Gamma(\mathbf{z})=\varnothing$ if and only if $\mathbf{z}=\mathbf{0}$, and that $\Gamma(\mathbf{z})=\langle n\rangle$ if and only if $z_i\neq 0$ for each $i\in\langle n\rangle$. When confusion is unlikely to occur we shall write Γ, Γ_+, and Γ_- instead of $\Gamma(\mathbf{z})$, $\Gamma_+(\mathbf{z})$, and $\Gamma_-(\mathbf{z})$, respectively.

In economic terms $\Gamma(\Delta\mathbf{c})$, $\Gamma_+(\Delta\mathbf{c})$, and $\Gamma_-(\Delta\mathbf{c})$ are those industries in which the demand changes, increases, or decreases, respectively.

THEOREM 6. *Let $r<s$. Then the following assertions hold:*

(a) $\Delta\mathbf{x}\gg\mathbf{0}\Leftrightarrow\Delta x_j>0$ *for each* $j\in\Gamma_-(\Delta\mathbf{c})$;
(b) $\Delta\mathbf{x}\ll\mathbf{0}\Leftrightarrow\Delta x_j<0$ *for each* $j\in\Gamma_+(\Delta\mathbf{c})$.

Proof. It is clear that (b) is to be shown in the same way as (a), so only the proof of (a) is given. In one direction the double implication of (a) is clear. So all we need to prove is that if $\Delta x_j>0$ for each $j\in\Gamma_-$, then $\Delta\mathbf{x}\gg\mathbf{0}$. Let T_* be the submatrix of T which arises from T by deleting all rows and columns with index in Γ_-. Note that the number of rows and columns of T^* is $|\Gamma_-|$. By I_* we understand the $(n-|\Gamma_-|,n-|\Gamma_-|)$ identity matrix; $\Delta\mathbf{x}_*$ is Δx with elements with index in Γ_- deleted; $\mathbf{1}_*$ and $\mathbf{0}_*$ are $\mathbf{1}$ and $\mathbf{0}$, respectively, with $n-|\Gamma_-|$ elements deleted. Obviously, the implication holds if $\Gamma_-=\varnothing$ and if $\Gamma_-=\langle n\rangle$. So we may assume that $\varnothing\neq\Gamma_-\neq\langle n\rangle$. Note that $0<n-|\Gamma_-|<n$. Let α be the smallest positive number such that $t_{ij}^{(\alpha)}>0$ for some $i\notin\Gamma_-$ and some $j\in\Gamma_-$. The definition of α implies directly that $t_{ij}^{(k)}=0$ for each $k\in\langle\alpha-1\rangle$, each $i\notin\Gamma_-$, and each $j\in\Gamma_-$. Note that if $t_{ij}>0$ for some $i\notin\Gamma_-$ and $j\in\Gamma_-$, then $\alpha=1$, because $t_{ij}^{(0)}=0$ for each $i,j\in\langle n\rangle$ with $i\neq j$. Hence, for each $k\in\langle\alpha-1\rangle$ all elements of the vector $T^k\Delta\mathbf{c}$ with index not in Γ_- are non-negative, and it follows from (9) that for each $m\notin\Gamma_-$

$$s^\alpha\Delta x_m-\sum_{k=1}^{n}t_{mk}^{(\alpha)}\Delta x_k\geqslant 0,$$

or

$$s^\alpha\Delta x_m-\sum_{k\notin\Gamma_-}t_{mk}^{(\alpha)}\Delta x_k\geqslant\sum_{k\in\Gamma_-}t_{mk}^{(\alpha)}\Delta x_k.$$

By definition, there is an element $m_o\notin\Gamma_-$ and an element $k_o\in\Gamma_-$ such that $t_{m_ok_o}^{(\alpha)}>0$, and as $\Delta x_k>0$ for each $k\in\Gamma_-$, it follows that $\sum_{k\in\Gamma_-}t_{m_ok}^{(\alpha)}\Delta x_k>0$. Hence,

$$\sum_{k\notin\Gamma_-}t_{m_ok}^{(\alpha)}\Delta x_k<s^\alpha\Delta x_{m_o}.$$

Therefore, $(s^\alpha I_*-T^\alpha{}_*)\Delta\mathbf{x}_*>\mathbf{0}_*$.

As $r<s$, it follows that the Perron-Frobenius eigenvalue of $T^{\alpha}{}_{*}$ is less than s^{α}, and according to Proposition 2 we find that $\Delta \mathbf{x}_{*} \gg \mathbf{0}_{*}$. Hence, $\Delta x_m > 0$ for each $m \notin \Gamma_{-}$, and so we have in fact that $\Delta \mathbf{x} \gg 0$. ∎

In terms of the open Leontief model Theorem 6 means that if the production increases of all commodities for which the demand decreases, then all productions increase, and if the production decreases of all commodities for which the demand increases, then all productions decrease. Note that if the production increases of an industry in which the demand decreases, then—by virtue of Theorem 3—there must be an industry in which the demand increases.

$\Delta x_j > 0$ does not necessarily imply that $\Delta c_j > 0$ for some $j \in \langle n \rangle$. So one may have the situation on the right-hand side of the implication in Theorem 6(a). Consider for instance the following matrix:

$$T = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix},$$

and take $s=4$. Then

$$(4I-T)^{-1} = \frac{1}{3}\begin{bmatrix} 5 & 3 & 4 \\ 3 & 3 & 3 \\ 4 & 3 & 5 \end{bmatrix}.$$

Taking $\Delta \mathbf{c} = (2, -1, 0)'$, it follows that $\Delta \mathbf{x} = \frac{1}{3}(7,3,5)' \gg \mathbf{0}$.

8. BOUNDS TO THE CHANGES OF THE PRODUCTIONS

THEOREM 7. *Let $(sI-T)\mathbf{1} > \mathbf{0}$. Then for each $i \in \langle n \rangle$ the following holds*:

$$\min\Big\{0, \inf_{j \in \Gamma_{-}} \Delta x_j\Big\} \leqslant \Delta x_i \leqslant \max\Big\{0, \sup_{j \in \Gamma_{+}} \Delta x_j\Big\}. \tag{10}$$

Proof. We shall only give the proof of the right-hand-side part of (10), for the proof of the other part is obviously analogous. The right-hand-side part of (10) is denoted by (10′).

Let $\Gamma_{+} = \varnothing$. Then $\max\{0, \sup_{j \in \Gamma_{+}} \Delta x_j\} = \max\{0, \sup\{\Delta x_j | j \in \varnothing\}\} = \max\{0, \sup \varnothing\} = \max\{0, -\infty\} = 0$. If, moreover, $\Gamma_{-} = \varnothing$, then $\Gamma = \varnothing$, and it follows from Theorem 3(a) that $\Delta \mathbf{x} = \mathbf{0}$, whence we may conclude that (10′) holds in case $\Gamma = \varnothing$. Now let $\Gamma_{-} \neq \varnothing$. Then $\Delta \mathbf{c} < \mathbf{0}$, and Theorem 3(c) implies that $\Delta\mathbf{x} \ll \mathbf{0}$. Hence, (10′) also holds in this case. So we may assume now that $\Gamma_+ \neq \varnothing$. This implies that we must show that for each $i \in \langle n \rangle$

$$\Delta x_i \leqslant \max\left\{0, \max_{j \in \Gamma_+} \Delta x_j\right\}. \tag{11}$$

If $\Delta x_j < 0$ for each $j \in \Gamma_+$, then $\max\{0, \max_{j \in \Gamma_+} \Delta x_j\} = 0$. On the other hand, Theorem 6 implies that $\Delta\mathbf{x} \ll \mathbf{0}$. Therefore, (11) holds in case $\Delta x_j < 0$ for each $j \in \Gamma_+$. So we may also assume now that $\Delta x_j \geqslant 0$ for some $j \in \Gamma_+$. Then (11) implies that we must show that for each $i \in \langle n \rangle$

$$\Delta x_i \leqslant \max_{j \in \Gamma_+} \Delta x_j.$$

To that end, let us suppose to the contrary that there is an element $i \notin \Gamma_+$ such that $\Delta x_i > \max_{j \in \Gamma_+} \Delta x_j$. Let $\Delta x_m = \max_{i \notin \Gamma_+} \Delta x_i$ with $m \notin \Gamma_+$. Hence,

$$\Delta x_m \begin{cases} \geqslant \Delta x_i & \text{if} \quad i \notin \Gamma_+, \\ > \Delta x_i & \text{if} \quad i \in \Gamma_+. \end{cases}$$

Note that $\Delta x_m > 0$. Let α be the smallest positive number such that $t_{mj}^{(\alpha)} > 0$ for some $j \in \Gamma_+$. This definition implies directly that $t_{mj}^{(k)} = 0$ for each $k \in \langle \alpha - 1 \rangle$, $\alpha \geqslant 2$, and each $j \in \Gamma_+$. Note that if $t_{mj} > 0$ for some $j \in \Gamma_+$ and $j \neq m$, then $\alpha = 1$, because $t_{mj}^{(0)} = 0$ for each $j \in \langle n \rangle$ with $j \neq m$. Lemma 5 implies that

$$(s^\alpha I - T^\alpha)\Delta\mathbf{x} = \sum_{k=1}^{\alpha} s^{\alpha-k} T^{k-1} \Delta\mathbf{c}.$$

Obviously, the mth element of the vector $T^{k-1}\Delta\mathbf{c}$ is nonpositive for each $k \in \langle \alpha \rangle$, so that

$$s^\alpha \Delta x_m \leqslant \sum_{k=1}^{n} t_{mk}^{(\alpha)} \Delta x_k < \Delta x_m \sum_{k=1}^{n} t_{mk}^{(\alpha)}.$$

As $\Delta x_m > 0$, it follows that $\sum_{k=1}^{n} t_{mk}^{(\alpha)} > s^\alpha$. On the other hand, it follows from Lemma 4(a) that $(s^\alpha I - T^\alpha)\mathbf{1} \geqslant \mathbf{0}$, and hence that $\sum_{k=1}^{n} t_{mk}^{(\alpha)} \leqslant s^\alpha$, and we arrive at a contradiction.

The conclusion is that in fact $\Delta x_i \leqslant \max_{j \in \Gamma_+} \Delta x_j$, and therefore (10′) holds. ∎

COROLLARY 8. *Let* $(sI-T)\mathbf{1}>\mathbf{0}$ *and* $m\in\langle n\rangle$. *Then*

(a) $\Delta x_m=\max_{j\in\Gamma_+}\Delta x_j\geqslant 0 \Rightarrow \Delta c_m\geqslant 0$;
(b) $\Delta x_m=\min_{j\in\Gamma_-}\Delta x_j\leqslant 0 \Rightarrow \Delta c_m\leqslant 0$.

Proof. We shall give the proof of (a). Let $\Delta x_m=\max_{j\in\Gamma_+}\Delta x_j\geqslant 0$ for some $m\in\langle n\rangle$. Then Theorem 7 implies that $\Delta x_m=\max_{i\in\langle n\rangle}\Delta x_i$. As $\Delta\mathbf{c}=s\,\Delta\mathbf{x}-T\Delta\mathbf{x}$, we find that $\Delta c_m=s\,\Delta x_m-\sum_{k=1}^{n}t_{mk}\Delta x_k\geqslant[s-\sum_{k=1}^{n}t_{mk}]\Delta x_m\geqslant 0$. ∎

Note that if $m\in\Gamma_+$ in Corollary 8(a), then $\Delta c_m>0$.

In terms of the open Leontief model Corollary 8(a) asserts that the maximum of all the increases of the productions occurs in an industry in which the demand does not decrease. The next example will demonstrate that Δc_m can be zero.

Take T as in the previous example, and take $s=4$. Then $\Delta\mathbf{c}=(0,1,0)'$ implies that $\Delta\mathbf{x}=(1,1,1)'$. Note that Δx_1 is the maximum, but $1\notin\Gamma_+=\{2\}$.

COROLLARY 9. *Let* $(sI-T)\mathbf{1}>\mathbf{0}$, *and let there be indices* $j_1,j_2\in\Gamma(\Delta\mathbf{c})$ *such that* $\Delta x_{j_1}<0$ *and* $\Delta x_{j_2}>0$. *Then for each* $i\in\langle n\rangle$ *the following holds.*

$$\min_{j\in\Gamma_-}\Delta x_j\leqslant\Delta x_i\leqslant\max_{j\in\Gamma_+}\Delta x_j. \tag{12}$$

Proof. Only the right-hand side of (12) is proved. First note that $\Gamma_+\neq\varnothing$. For if $\Gamma_+=\varnothing$, then $\Delta\mathbf{c}\leqslant\mathbf{0}$, and it follows from Theorem 3(a), (c) that $\Delta\mathbf{x}<\mathbf{0}$, which contradicts the fact that $\Delta x_{j_2}>0$. Now Theorem 7 implies that for each $i\in\langle n\rangle$, $\Delta x_i\leqslant\max\{0,\sup_{j\in\Gamma_+}\Delta x_j\}=\max_{j\in\Gamma_+}\Delta x_j$. ∎

In terms of the open Leontief model Corollary 9 asserts that the maximal increase or minimal decrease of the productions occurs in those industries in which the demand increases or decreases, respectively.

The following examples show that the bounds in (10) are sharp.

Take T as in the previous example and $s=4$. Let $\Delta\mathbf{c}=(4,0,-1)'$. Then $\Gamma_+=\{1\}$ and $\Gamma_-=\{3\}$. It is easy to verify that $\Delta\mathbf{x}=\frac{1}{3}(16,9,11)$. Hence, $\max_{j\in\Gamma_+}\Delta x_j=\Delta x_1=\frac{16}{3}$ and $\min_{j\in\Gamma_-}\Delta x_j=\Delta x_3=\frac{11}{3}$. Note that $\Delta x_2<\min_{j\in\Gamma_-}\Delta x_j$.

Taking $\Delta\mathbf{c}=(1,-1,0)'$, it follows that $\Delta\mathbf{x}=\frac{1}{3}(2,0,1)'$. Hence, $\min_{j\in\Gamma_-}\Delta x_j=0$. Now let $\Delta\mathbf{c}=(0,1,0)'$. Then $\Delta\mathbf{x}=(1,1,1)'$. So it may happen that the bounds in Theorem 7 are achieved in case $i\notin\Gamma$. Theorem 6 shows that the lower bound in (10) is not achieved if $\Delta x_j>0$ for each $j\in\Gamma_-$. Also, if $(sI-T)\mathbf{1}>0$ and $\Delta\mathbf{c}>\mathbf{0}$, then—by Theorem 3(b)—the lower bound is not achieved.

THEOREM 10. *Let* $T \gg 0$, $(sI - T)\mathbf{1} \gg \mathbf{0}$, *and* $\Gamma(\Delta\mathbf{c}) \neq \varnothing$. *Then for each* $i \notin \Gamma(\Delta\mathbf{c})$ *the following holds*:

$$\min\left\{0, \inf_{j \in \Gamma_-} \Delta x_j\right\} < \Delta x_i < \max\left\{0, \sup_{j \in \Gamma_+} \Delta x_j\right\}. \tag{13}$$

Proof. We shall only give the proof of the right-hand side of (13). Clearly, if $\Gamma_+ = \varnothing$, then $\Delta\mathbf{c} < \mathbf{0}$, and it follows from Theorem 3(c) that $\Delta\mathbf{x} \ll \mathbf{0}$. Hence, $\Delta x_i < 0 = \max\{0, \sup_{j \in \varnothing} \Delta x_j\}$ for each $i \in \langle n \rangle$. So we may assume that $\Gamma_+ \neq \varnothing$. If $\Delta x_j < 0$ for each $j \in \Gamma_+$, then—again by Theorem 6(b)—$\Delta\mathbf{x} \ll \mathbf{0}$, which is what we set out to prove. Therefore, we may assume that $\Delta x_j \geqslant 0$ for some $j \in \Gamma_+$. Now suppose, to the contrary, that for some $i \notin \Gamma_+$, $\Delta x_i \geqslant \max_{j \in \Gamma_+} \Delta x_j$. Let $\Delta x_m = \max_{i \notin \Gamma_+} \Delta x_i$ for some $m \notin \Gamma_+$. Hence, $\Delta c_m \leqslant 0$, and it follows that $\Delta x_m \geqslant \Delta x_i$ for each $i \in \langle n \rangle$. As $T \gg 0$, it follows from $(sI - T)\Delta\mathbf{x} = \Delta\mathbf{c}$ that $s\Delta x_m \leqslant \sum_{k=1}^n t_{mk} \Delta x_k \leqslant \Delta x_m \sum_{k=1}^n t_{mk}$. If $\Delta x_m = 0$, then $\sum_{k=1}^n t_{mk} \Delta x_k = 0$. This implies, as $T \gg 0$, that $\Delta\mathbf{x} = \mathbf{0}$, but then—by Theorem 3(a)—$\Delta\mathbf{c} = 0$, which implies that $\Gamma = \varnothing$, and this is a contradiction. Hence, $\Delta x_m > 0$. So we find that $\sum_{k=1}^n t_{mk} \geqslant s$. However, $(sI - T)\mathbf{1} \gg \mathbf{0}$ implies that $\sum_{k=1}^n t_{ik} < s$ for each $i \in \langle n \rangle$. So we have another contradiction.

The conclusion is that in fact $\Delta x_i < \max_{j \in \Gamma_+} \Delta x_j$ for each $i \notin \Gamma(\Delta\mathbf{c})$. ∎

The right-hand side of (13) holds for $i \notin \Gamma_+(\Delta\mathbf{c})$, and the left-hand side for $i \notin \Gamma_-(\Delta\mathbf{c})$.

In terms of the open Leontief model this theorem asserts that for any positive matrix with all row sums strictly less than unity, the changes of the productions for industries in which the demands remain constant are strictly between the maximal increase or minimal decrease of the productions of those industries in which the demands increase or decrease, respectively.

9. THE METZLER THEOREM

The symbol $c_{ij}(s)$ stands for the (i, j) cofactor of the matrix $sI - T$, and $\operatorname{adj}(sI - T) = \{c_{ji}\}$ stands for the *adjoint* of $sI - T$. Note that $\operatorname{adj}(sI - T)$ is the transpose of the matrix of the cofactors of $sI - T$. When there can't be any ambiguity we shall write c_{ij} instead of $c_{ij}(s)$. Proposition 2 implies directly that the inverse of $sI - T$ exists and that $(sI - T)^{-1} \gg 0$, provided that $r < s$. Recall that if $r < s$, then

$$(sI - T)^{-1} = \frac{\operatorname{adj}(sI - T)}{|sI - T|}. \tag{14}$$

So if $r<s$, then it follows from $(sI-T)\mathbf{x}=\mathbf{c}$ that $\mathbf{x}=(sI-T)^{-1}\mathbf{c}=(1/|sI-T|)\operatorname{adj}(sI-T)\mathbf{c}$; hence for each $i\in\langle n\rangle$ we find that

$$x_i = \frac{1}{|sI-T|}\sum_{k=1}^{n} c_{ki}c_k. \tag{15}$$

LEMMA 11. *If $(sI-T)\mathbf{1}\geqslant \mathbf{0}$, then the following assertions are equivalent*:

(a) $(sI-T)\mathbf{1}=\mathbf{0}$ *(all row sums of T are equal to s)*;
(b) $|sI-T|=0$;
(c) $\operatorname{adj}(sI-T)\gg 0$ *and* $c_{ki}=c_{kj}$ *for each* $k,i,j\in\langle n\rangle$ *(all rows of* $\operatorname{adj}(sI-T)$ *are equal)*.

Proof.

(a)⇒(b): $(sI-T)\mathbf{1}=\mathbf{0}$ implies that the system $(sI-T)\mathbf{x}=\mathbf{0}$ of homogeneous equations has a nonzero solution, namely $\mathbf{1}$; hence $|sI-T|=0$.

(b)⇒(a): If, to the contrary, $(sI-T)\mathbf{1}>\mathbf{0}$, then—by Proposition 1—$r<s$, so $|sI-T|>0$. This is a contradiction. Therefore, we have that $(sI-T)\mathbf{1}=\mathbf{0}$.

(b)⇒(c): The following identity is well known (see e.g. [5], p. 5):

$$(sI-T)\operatorname{adj}(sI-T) = |sI-T|I. \tag{16}$$

As $|sI-T|=0$, (16) implies that $(sI-T)\operatorname{adj}(sI-T)\mathbf{e}_i=\mathbf{0}$ for each $i\in\langle n\rangle$. As (a)⇔(b), we find that for each $i\in\langle n\rangle$ there is an element $\alpha_i\neq 0$ such that

$$\operatorname{adj}(sI-T)\mathbf{e}_i = \alpha_i\mathbf{1},$$

and this implies that all rows of $\operatorname{adj}(sI-T)$ are equal. Proposition 1 together with (a) implies that $r=s$, and therefore $\operatorname{adj}(sI-T)=\operatorname{adj}(rI-T)>0$; see e.g. [5], p. 5. Hence, $c_{ij}>0$ for each $i,j\in\langle n\rangle$.

(c)⇒(a): As all rows of $\operatorname{adj}(sI-T)$ are equal, it follows that $|\operatorname{adj}(sI-T)|=0$. If, to the contrary, $(sI-T)>\mathbf{0}$, then—by Proposition 1—$r<s$, and (16) implies that $|\operatorname{adj}(sI-T)|=1$, so we have a contradiction. Therefore, $(sI-T)\mathbf{1}=\mathbf{0}$. ■

The following theorem is the same as [5, Theorem 2.3]. The proofs are quite different, however; our proof is a direct consequence of Theorem 7 and Lemma 11. See also [4], p. 18.

THEOREM 12 (Metzler's theorem). *If* $(sI-T)\mathbf{1}\geqslant\mathbf{0}$, *then for each* $i,j\in\langle n\rangle$ *the following holds*:

$$0 < c_{ij}(s) \leqslant c_{ii}(s). \tag{17}$$

Proof. We distinguish the following two cases:

(a) $(sI-T)>\mathbf{0}$. Take any $\Delta\mathbf{c}>\mathbf{0}$ such that $\Gamma=\Gamma_+=\{i\}$ for some $i\in\langle n\rangle$. Theorem 3(b) implies that $\Delta\mathbf{x}\gg\mathbf{0}$, and Theorem 7 that $\Delta x_j\leqslant\Delta x_i$ for each $j\in\langle n\rangle$. Now it follows from (15) that

$$0 < \frac{1}{|sI-T|}c_{ij}\Delta c_i \leqslant \frac{1}{|sI-T|}c_{ii}\Delta c_i,$$

so that in fact $0<c_{ij}\leqslant c_{ii}$ for each $i,j\in\langle n\rangle$.

(b) $(sI-T)\mathbf{1}=\mathbf{0}$. Lemma 11(c) now implies directly that $0<c_{ij}=c_{ii}$ for each $i,j\in\langle n\rangle$.

From (a) and (b) it follows that (17) is true. ∎

THEOREM 13. *If* $T\gg 0$ *and* $(sI-T)\mathbf{1}\gg\mathbf{0}$, *then for each* $i,j\in\langle n\rangle$ *with* $i\neq j$ *the following holds:*

$$0 < c_{ij}(s) < c_{ii}(s). \tag{18}$$

Proof. Take any $\Delta\mathbf{c}>\mathbf{0}$ such that $\Gamma=\Gamma_+=\{i\}$ for some $i\in\langle n\rangle$. Then Theorem 3(b) implies that $\Delta\mathbf{x}\gg\mathbf{0}$, and Theorem 10 that $\Delta x_j<\Delta x_i$ for each $j\in\langle n\rangle$ with $j\neq i$. From (15) it follows that in fact $0<c_{ij}<c_{ii}$ for each $i,j\in\langle n\rangle$ with $i\neq j$. ∎

In [5] Seneta asserts that $c_{ij}<c_{ii}$ for each $i,j\in\langle n\rangle$ provided $T>0$ (Corollary 1 of Theorem 2.3). In his formulation two requirements should be added, viz. that $i\neq j$ and that all row sums must be strictly less than s. Theorem 13 can also be found in G. Debreu and I. N. Herstein [2, p. 603].

10. MATRICES OF CLASS $M(x)$

Let $x\in\mathbb{R}$. We say that an (n,n) matrix $A=\{a_{ij}\}$ is of *class* $M(x)$ if and only if

$$a_{jj}-a_{ij}=x$$

for each $i,j\in\langle n\rangle$ with $i\neq j$.

This definition implies that if $A=\{a_{ij}\}\in M(x)$, then

$$A=\begin{bmatrix} a_{11} & a_{22}-x & \cdots & a_{nn}-x \\ a_{11}-x & a_{22} & \cdots & a_{nn}-x \\ a_{11}-x & a_{22}-x & \cdots & a_{nn}-x \\ \vdots & \vdots & \ddots & \vdots \\ a_{11}-x & a_{22}-x & \cdots & a_{nn} \end{bmatrix}.$$

Examples of matrices of class $M(x)$ are therefore:

$$\begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix},\quad \begin{bmatrix} 5 & 2 & 3 \\ 1 & 6 & 3 \\ 1 & 2 & 7 \end{bmatrix},\quad \begin{bmatrix} 2 & 1 & 2 & 0 \\ 4 & -1 & 2 & 0 \\ 4 & 1 & 0 & 0 \\ 4 & 1 & 2 & -2 \end{bmatrix}.$$

Note that $I\in M(1)$.

LEMMA 14. *Let $x\in\mathbb{R}$. Then $A\in M(x)\Leftrightarrow sI-A\in M(s-x)$.*

Proof. $A\in M(x)\Leftrightarrow a_{jj}-a_{ij}=x,\ i\neq j\Leftrightarrow (s-a_{jj})-(-a_{ij})=s-x,\ i\neq j\Leftrightarrow sI-A\in M(s-x)$. ∎

LEMMA 15. *Let $A=\{a_{ij}\}\in M(x)$ for some $x\in\mathbb{R}$. Then $A\geqslant 0$, irreducible $\Leftrightarrow x<a_{ii}$ and $a_{ii}\geqslant 0$ for each $i\in\langle n\rangle$.*

Proof. Trivial. ∎

THEOREM 16. *Let $x\in\mathbb{R}$. Then $A=\{a_{ij}\}\in M(x)$ implies that*

$$|A| = x^{n-1}\left(\sum_{i=1}^{n} a_{ii}-(n-1)x\right).$$

Proof.

$$|A|=\begin{vmatrix} a_{11} & a_{22}-x & \cdots & a_{nn}-x \\ a_{11}-x & a_{22} & \cdots & a_{nn}-x \\ \vdots & \vdots & \ddots & \vdots \\ a_{11}-x & a_{22}-x & \cdots & a_{nn} \end{vmatrix}=\begin{vmatrix} a_{11} & a_{22}-x & \cdots & a_{nn}-x \\ -x & x & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -x & 0 & \cdots & x \end{vmatrix}$$

$$= a_{11}x^{n-1} + x^{n-1}(a_{22} - x) + \cdots + x^{n-1}(a_{nn} - x)$$

$$= x^{n-1}\left(\sum_{i=1}^{n} a_{ii} - (n-1)x \right),$$

where we have expanded the right-hand determinant on its first column. ■

THEOREM 17. *Let $x \in \mathbb{R}$ and $A \in M(x)$. Then the following holds*:

$$x^{n-1}A\mathbf{1} = |A|\mathbf{1}.$$

Proof. Clearly, $A\mathbf{1} = \sum_{i=1}^{n} a_{ii} - (n-1)x$. Theorem 16 implies directly that $x^{n-1}A\mathbf{1} = |A|\mathbf{1}$. ■

Note that all row sums of class-$M(x)$ matrices are equal.

THEOREM 18. *Let $x \in \mathbb{R} \backslash \{0\}$. Then*

$$A \in M(x) \quad \Leftrightarrow \quad \operatorname{adj}(A) \in M\left(\frac{|A|}{x} \right).$$

Proof. Take any $A \in M(x)$ with $x \in \mathbb{R}\backslash\{0\}$, and let $\operatorname{adj}(A) = \{b_{ij}\}$. It follows directly from Theorem 16 that $b_{jj} = x^{n-2}[\sum_{k \neq j} a_{kk} - (n-2)x]$ for each $j \in \langle n \rangle$. Obviously, $b_{ij} = -x^{n-2}(a_{jj} - x)$ for each $i, j \in \langle n \rangle$ with $i \neq j$. Hence, $b_{jj} - b_{ij} = x^{n-2}(\sum_{k \neq j} a_{kk} - nx + 2x + a_{jj} - x) = x^{n-2}[\sum_{k=1}^{n} a_{kk} - (n-1)x] = |A|/x$. The reverse implication follows analogously. ■

11. MAXIMAL AND MINIMAL CHANGE OF DEMAND AND PRODUCTION

THEOREM 19. *Let $(sI - T)\mathbf{1} > \mathbf{0}$. Then the following assertions are equivalent*:

(i) $T \in M(x)$ *for some* $x \in \mathbb{R}$;
(ii) $\Delta x_i < \Delta x_j \Leftrightarrow \Delta c_i < \Delta c_j$, *for each* $i, j \in \Gamma(\Delta\mathrm{c})$.

Proof.

(i)$\Rightarrow$(ii): Take any $x \in \mathbb{R}$ and any $T \in M(x)$. As all row sums of T are equal (see Theorem 17), it follows that for each $i \in \langle n \rangle$, $\sum_{j=1}^{n} t_{ij} = \sum_{k=1}^{n} t_{kk} -$

$(n-1)x<s$. As $t_{ii}-x>0$ for each $i\in\langle n\rangle$ (see Lemma 15), it follows that

$$s > \sum_{k=1}^{n} t_{kk} - (n-1)x > nx - (n-1)x = x.$$

Take any $i,j\in\Gamma(\Delta c)$ with $i\neq j$. Then

$$\begin{aligned}\Delta c_i < \Delta c_j \quad \Leftrightarrow \quad & -\sum_{k\neq i,j} (t_{kk}-x)\Delta x_k + (s-t_{ii})\Delta x_i - (t_{jj}-x)\Delta x_j \\ & < -\sum_{k\neq i,j} (t_{kk}-x)\Delta x_k + (s-t_{jj})\Delta x_j - (t_{ii}-x)\Delta x_i \\ \Leftrightarrow \quad & (s-t_{ii})\Delta x_i - (t_{jj}-x)\Delta x_j < (s-t_{jj})\Delta x_j - (t_{ii}-x)\Delta x_i \\ \Leftrightarrow \quad & s\Delta x_i - t_{ii}\Delta x_i - t_{jj}\Delta x_j + x\Delta x_j < s\Delta x_j - t_{jj}\Delta x_j - t_{ii}\Delta x_i + x\Delta x_i \\ \Leftrightarrow \quad & s\Delta x_i + x\Delta x_j < s\Delta x_j + x\Delta x_i \\ \Leftrightarrow \quad & (s-x)\Delta x_i < (s-x)\Delta x_j \\ \Leftrightarrow \quad & \Delta x_i < \Delta x_j.\end{aligned}$$

(ii)⇒(i): Suppose, to the contrary, that $T\notin M(x)$ for each $x\in\mathbb{R}$. We distinguish two cases:

(a) $t_{\alpha j}\neq t_{\beta j}$ for some $\alpha,\beta,j\in\langle n\rangle$ with $\alpha\neq j\neq\beta\neq\alpha$. So there are elements $x,y\in\mathbb{R}$ such that $t_{\alpha j}=t_{jj}-x\neq t_{jj}-y=t_{\beta j}$. Let $x<y$, and take Δx such that $\Delta x_j=1$, $\Delta x_\alpha=(y-x)/(s-t_{\alpha\alpha}+t_{\beta\alpha})>0$, and $\Delta x_i=0$, with $\alpha\neq i\neq j$. Then it follows that $\Delta c_\alpha=-t_{jj}+x+(s-t_{\alpha\alpha})\Delta x_\alpha$ and $\Delta c_\beta=-t_{jj}+y-t_{\beta\alpha}\Delta x_\alpha$. As $\Delta x_\alpha>\Delta x_\beta=0$, it follows from (ii) that $\Delta c_\alpha>\Delta c_\beta$. Hence, $-t_{jj}+x+s\Delta x_\alpha-t_{\alpha\alpha}\Delta x_\alpha>-t_{jj}+y-t_{\beta\alpha}\Delta x_\alpha$, or $y-x<(s-t_{\alpha\alpha}+t_{\beta\alpha})\Delta x_\alpha=y-x$. This is a contradiction, so we have that $x=y$, or that $t_{\alpha j}=t_{\beta j}$ for each $\alpha,\beta,j\in\langle n\rangle$ with $\alpha\neq j\neq\beta$.

(b) $t_{\alpha\alpha}-t_{i\alpha}\neq t_{\beta\beta}-t_{j\beta}$ for some $i,j,\alpha,\beta\in\langle n\rangle$ with $i\neq\alpha\neq\beta\neq j$. So there are elements $x,y\in\mathbb{R}$ such that $t_{\alpha\alpha}-x\neq t_{\beta\beta}-y$. Let $x<y$, and take Δx such that $\Delta x_\alpha\neq0\neq\Delta x_\beta$ and $\Delta x_i=0$ for each $i\neq\alpha,\beta$. Then

$$\begin{aligned}\Delta c_\alpha-\Delta c_\beta &= \left[(s-t_{\alpha\alpha})\Delta x_\alpha+(-t_{\beta\beta}+y)\Delta x_\beta\right]-\left[(-t_{\alpha\alpha}+x)\Delta x_\alpha+(s-t_{\beta\beta})\Delta x_\beta\right] \\ &= (s-x)\Delta x_\alpha-(s-y)\Delta x_\beta.\end{aligned}$$

Note that $x<y$ implies that $s-x>s-y$. First let $s-x=0$. Then $s-y<0$. Taking $\Delta x_\beta=2\Delta x_\alpha>0$, we find that $\Delta x_\alpha<\Delta x_\beta$, but that $\Delta c_\alpha>\Delta c_\beta$. This contradicts (ii). In the same way, $s-y=0$ leads to a contradiction. So we may assume that $s-x\neq0\neq s-y$. Taking $\Delta x_\beta=[(s-x)/(s-y)]\Delta x_\alpha$, we find that $\Delta x_\beta > \Delta x_\alpha$, but that $\Delta c_\alpha = \Delta c_\beta$, which contradicts (ii). Therefore, $x = y$, and thus $t_{\alpha\alpha} - t_{i\alpha} = t_{\beta\beta} - t_{j\beta}$ for each $i, j, \alpha, \beta \in \langle n \rangle$ with $i \neq \alpha \neq \beta \neq j$.

The conclusion is that in fact $T \in M(x)$ for some $x \in \mathbb{R}$. ■

COROLLARY 20. *Let* $(sI - T)1 > 0$ *and* $\Gamma(\Delta c) \neq \varnothing$. *Then* $T \in M(x)$ *for some* $x \in \mathbb{R}$ *if and only if one of the following assertions holds*:

(a) $\Delta x_m = \max_{i \in \langle n \rangle} \Delta x_i \Leftrightarrow \Delta c_m = \max_{j \in \Gamma} \Delta c_j$;

(b) $\Delta x_m = \min_{i \in \langle n \rangle} \Delta x_i \Leftrightarrow \Delta c_m = \min_{j \in \Gamma} \Delta c_j$.

Proof. It follows directly from Theorem 19(ii) that if $T \in M(x)$ for some $x \in \mathbb{R}$, then (a) holds. Now suppose that (a) holds. We'll show that $T \in M(x)$ for some $x \in \mathbb{R}$, or, equivalently, that Theorem 19(ii) holds. Suppose, to the contrary, that there are vectors Δc and Δx such that $\Delta x_i < \Delta x_j$ and $\Delta c_i \geqslant \Delta c_j$ for some $i, j \in \langle n \rangle$. Let Δc be the zero vector except that the ith and jth elements are Δc_i and Δc_j, respectively. It then follows that Δc_i is the maximum, and by (a) we find that Δx_i is the maximum. But this contradicts $\Delta x_i < \Delta x_j$. Therefore we have that in fact Theorem 19(ii) holds. ■

In terms of the open Leontief model, Corollary 20 asserts that the maximal or minimal change of the demand and the production occurs in the same industry if and only if the matrix T is of class $M(x)$ for some $x \in \mathbb{R}$. So —as was to be expected—if the maximal or minimal change of the demand occurs in one industry, then the maximal or minimal change of the production will generally occur in another. Consider the following example with $T \in M(-2)$: Take

$$T = \begin{bmatrix} 0 & 3 & 6 \\ 2 & 1 & 6 \\ 2 & 3 & 4 \end{bmatrix}.$$

Note that all row sums are equal to 9. Take $s = 10$. Then

$$10I - T = \begin{bmatrix} 10 & -3 & -6 \\ -2 & 9 & -6 \\ -2 & -3 & 6 \end{bmatrix} \quad \text{and} \quad (10I - T)^{-1} = \frac{1}{144}\begin{bmatrix} 36 & 36 & 72 \\ 24 & 48 & 72 \\ 24 & 36 & 84 \end{bmatrix}.$$

Taking $\Delta c = (12, -12, 24)'$, it follows that $\Delta x = (12, 10, 13)'$. Note that $\Delta c_3 = \max_{i \in \langle 3 \rangle} \Delta c_i$, and $\Delta x_3 = \max_{i \in \langle 3 \rangle} \Delta x_i$, and that $\Delta c_2 = \min_{i \in \langle 3 \rangle} \Delta c_i$, and $\Delta x_2 = \min_{i \in \langle 3 \rangle} \Delta x_i$.

12. BOUNDS TO THE RELATIVE CHANGE OF DEMAND AND PRODUCTION

In the previous part of this paper only absolute changes of the final demand and the gross production were studied; the remainder will be devoted to the study of relative changes.

Let $c_i \neq 0$ and $x_i \neq 0$ for some $i \in \langle n \rangle$. The *relative change* of the final demand c_i and the gross production x_i of any commodity i are formulated as

$$\frac{\Delta x_i}{x_i} \quad \text{and} \quad \frac{\Delta c_i}{c_i},$$

respectively.

The next theorem is a generalization of M. Morishima's [4, Theorem 6], and the proof relies partly on a technique used in Morishima's proof.

THEOREM 21. *Let $r < s$ and $\mathbf{c} > \mathbf{0}$. Then for each $i \in \langle n \rangle$ the following holds*:

$$\min\left\{0, \inf_{j \in \Gamma_-} \frac{\Delta x_j}{x_j}\right\} \leqslant \frac{\Delta x_i}{x_i} \leqslant \max\left\{0, \sup_{j \in \Gamma_+} \frac{\Delta x_j}{x_j}\right\}. \tag{19}$$

Proof. Only the proof of the right-hand side of (19) is given. Note that Proposition 2 implies that $\mathbf{x} \gg \mathbf{0}$. If $\Gamma_+ = \varnothing$, then (19) holds trivially (see the proof of Theorem 7). So we may assume that $\Gamma_+ \neq \varnothing$. If $\Delta x_j < 0$ for each $j \in \Gamma_+$, then Theorem 6(b) implies that $\Delta \mathbf{x} \ll \mathbf{0}$. Hence, $\sup_{j \in \Gamma_+} \Delta x_j / x_j < 0$, so that in fact $\Delta x_i / x_i < 0 = \max\{0, \sup_{j \in \Gamma_+} \Delta x_j / x_j\}$ for each $i \in \langle n \rangle$. We may also assume, therefore, that $\Delta x_j \geqslant 0$ for some $j \in \Gamma_+$. It now follows that $\max\{0, \sup_{j \in \Gamma_+} \Delta x_j / x_j\} = \max_{j \in \Gamma_+} \Delta x_j / x_j$, so we just have to show that for each $i \in \langle n \rangle$,

$$\frac{\Delta x_i}{x_i} \leqslant \max_{j \in \Gamma_+} \frac{\Delta x_j}{x_j}. \tag{20}$$

Define $\mathbf{c}' = \mathbf{c} + \Delta \mathbf{c} > \mathbf{0}$ and $\mathbf{x}' = \mathbf{x} + \Delta \mathbf{x}$. For each $i \in \langle n \rangle$ there are elements $\mu_i, \lambda_i \in \mathbb{R}$ such that

$$c_i = \mu_i c_i' \quad \text{and} \quad x_i = \lambda_i x_i'.$$

Clearly, $\mu_j \neq 1$ for each $j \in \Gamma$, and $\lambda_i \neq 0$ for each $i \in \langle n \rangle$.

Lemma 5 assures that for each $\alpha \in \mathbb{N}$ the following holds:

$$(s^{\alpha} I - T^{\alpha})\mathbf{x} = \sum_{h=1}^{\alpha} s^{\alpha-h} T^{h-1} \mathbf{c},$$

so that for each $i \in \langle n \rangle$,

$$s^{\alpha} x_i = \sum_{k=1}^{n} t_{ik}^{(\alpha)} x_k + \sum_{k=1}^{n} \sum_{h=1}^{\alpha} s^{\alpha-h} t_{ik}^{(h-1)} c_k,$$

or

$$s^{\alpha} x_i' = \sum_{k=1}^{n} t_{ik}^{(\alpha)} \frac{\lambda_k}{\lambda_i} x_k' + \sum_{k=1}^{n} \sum_{h=1}^{\alpha} s^{\alpha-h} t_{ik}^{(h-1)} \frac{\mu_k}{\lambda_i} c_k'. \tag{21}$$

On the other hand, we have that $(sI - T)\mathbf{x}' = \mathbf{c}'$, so that for each $i \in \langle n \rangle$

$$s^{\alpha} x_i' = \sum_{k=1}^{n} t_{ik}^{(\alpha)} x_k' + \sum_{k=1}^{n} \sum_{h=1}^{\alpha} s^{\alpha-h} t_{ik}^{(h-1)} c_k'. \tag{22}$$

(21) and (22) imply that for each $\alpha \in \mathbb{N}$ and each $i \in \langle n \rangle$

$$\sum_{k=1}^{n} t_{ik}^{(\alpha)} \left(\frac{\lambda_k}{\lambda_i} - 1 \right) x_k' + \sum_{k=1}^{n} \sum_{h=1}^{\alpha} s^{\alpha-h} t_{ik}^{(h-1)} \left(\frac{\mu_k}{\lambda_i} - 1 \right) c_k' = 0. \tag{23}$$

Obviously (20) is equivalent to

$$\max_{i \in \langle n \rangle} \frac{\Delta x_i}{x_i} = \max_{j \in \Gamma_+} \frac{\Delta x_j}{x_j}. \tag{24}$$

Suppose to the contrary that

$$\max_{i \in \langle n \rangle} \frac{\Delta x_i}{x_i} > \max_{j \in \Gamma_+} \frac{\Delta x_j}{x_j} \geqslant 0. \tag{25}$$

Define $\Lambda \subset \langle n \rangle$ as follows:

$$m \in \Lambda \quad \Leftrightarrow \quad \max_{i \in \langle n \rangle} \frac{\Delta x_i}{x_i} = \frac{\Delta x_m}{x_m}. \tag{26}$$

Clearly, $\Lambda \cap \Gamma_+ = \varnothing$. As $\Delta x_i / x_i = (1/\lambda_i) - 1$ for each $i \in \langle n \rangle$, it follows from (25) and (26) that for each $i \in \Lambda$,

$$\frac{1}{\lambda_i} - 1 > \max_{j \in \Gamma_+} \left(\frac{1}{\lambda_j} - 1 \right) \geqslant 0, \tag{27}$$

or that

$$\lambda_i < \min_{j \in \Gamma_+} \lambda_j. \tag{28}$$

(26) implies that for each $m \in \Lambda$

$$\lambda_m = \min_{i \in \langle n \rangle} \lambda_i. \tag{29}$$

Now let β be the smallest positive integer such that

$$t_{i_o k_o}^{(\beta)} \neq 0$$

for some $i_o \in \Lambda$ and some $k_o \in \langle n \rangle \backslash \Lambda$. This definition implies directly that $t_{ik}^{(h-1)} = 0$ for each $h \in \langle \beta \rangle$, each $i \in \Lambda$, and each $k \notin \Lambda$. It follows from (23) that

$$\sum_{k=1}^{n} t_{i_o k}^{(\beta)} \left(\frac{\lambda_k}{\lambda_{i_o}} - 1 \right) x'_k + \sum_{k \in \Lambda} \sum_{h=1}^{\beta} s^{\beta - h} t_{i_o k}^{(h-1)} \left(\frac{\mu_k}{\lambda_{i_o}} - 1 \right) c'_k = 0. \tag{30}$$

Now $c'_k \geqslant 0$, $\lambda_{i_o} < 1$, and $\mu_k \geqslant 1$, so that $\dfrac{\mu_k}{\lambda_{i_o}} - 1 > 0$; thus (30) implies that

$$\sum_{k=1}^{n} t_{i_o k}^{(\beta)} \left(\frac{\lambda_k}{\lambda_{i_o}} - 1 \right) x'_k \leqslant 0. \tag{31}$$

(28) and (29) imply that $\lambda_{i_o} < \lambda_{k_o}$, or that

$$\frac{\lambda_{k_o}}{\lambda_{i_o}} - 1 > 0.$$

Hence $t_{i_o k_o}^{(\beta)} = 0$, and this contradicts the definition of β. Therefore we find that in fact (20) holds. ■

COROLLARY 22. *Let $r<s$ and $m \in \langle n \rangle$. Then*

(a) $\dfrac{\Delta x_m}{x_m} = \max\limits_{j \in \Gamma_+} \dfrac{\Delta x_j}{x_j} \geqslant 0 \Rightarrow \Delta c_m \geqslant 0;$

(b) $\dfrac{\Delta x_m}{x_m} = \min\limits_{j \in \Gamma_-} \dfrac{\Delta x_j}{x_j} \leqslant 0 \Rightarrow \Delta c_m \leqslant 0.$

Proof. We only give the proof of part (a). Let $\Delta x_m / x_m = \max_{j \in \Gamma_+} \Delta x_j / x_j \geqslant 0$ for some $m \in \langle n \rangle$. Then Theorem 21 implies that $\Delta x_m / x_m = \max_{i \in \langle n \rangle} \Delta x_i / x_i \geqslant 0$, and thus $\lambda_m = \min_{i \in \langle n \rangle} \lambda_i \leqslant 1$.. Taking $\alpha = 1$ in (23), we find that

$$\sum_{k=1}^{n} t_{mk}\left(\frac{\lambda_k}{\lambda_m} - 1\right)x'_k + \left(\frac{\mu_m}{\lambda_m} - 1\right)c'_m = 0.$$

As $(\lambda_k/\lambda_m) - 1 \geqslant 0$ and $t_{mk} \geqslant 0$ for each $k \in \langle n \rangle$, it follows that

$$[(\mu_m/\lambda_m) - 1]c'_m \leqslant 0.$$

As $c'_m \geqslant 0$, we find that $(\mu_m/\lambda_m) - 1 \leqslant 0$, or that $\mu_m \leqslant \lambda_m$. Hence, $\mu_m \leqslant 1$ and we therefore find that $c'_m \geqslant c_m$ or $\Delta c_m \geqslant 0$. ∎

Note that $m \notin \Gamma_-$ in Corollary 22(a) and $m \notin \Gamma_+$ in 22(b).

In terms of the open Leontief model Corollary 22(a) asserts that the maximum of all relative increases of the productions occurs in an industry in which the demand does not decrease. The following example shows that Δc_m may be zero.

Let T be the same as in the example at the end of Sec. 7, and let $s=4$. Take $\Delta \mathbf{c} = \mathbf{c} = (0,1,0)'$. Then $\Delta \mathbf{x} = \mathbf{x} = (1,1,1)'$. This implies that $\Delta x_1 / x_1$ is the maximum. Hence, $m=1$ and $\Delta c_m = 0$.

COROLLARY 23. *Let $r<s$, and let there be indices $j_1, j_2 \in \Gamma(\Delta \mathbf{c})$ such that $\Delta x_{j_1} < 0$ and $\Delta x_{j_2} > 0$. Then the following holds:*

$$\min_{j \in \Gamma_-} \frac{\Delta x_j}{x_j} \leqslant \frac{\Delta x_i}{x_i} \leqslant \max_{j \in \Gamma_+} \frac{\Delta x_j}{x_j}. \tag{32}$$

Proof. The proof is quite similar to the one of Corollary 9. ∎

In terms of the open Leontief model Corollary 23 asserts that the maximal relative increase or the minimal relative decrease of the productions occurs in those industries in which the demand increases or decreases, respectively.

It follows directly from Corollary 23 that if $\Gamma_+(\Delta\mathbf{c}) = \{j\}$ for some $j \in \langle n \rangle$, then $\Delta x_i / x_i \leqslant \Delta x_j / x_j$ for each $i \in \langle n \rangle$. So it is not necessary to replace the condition $r < 1$ by a stronger one as Seneta did in [5, p. 31, (2.5)].

The bounds in Theorem 21 are sharp. This follows by taking $\mathbf{x} = (1,1,1)'$ in the examples under Corollary 9.

COROLLARY 24. *Let $r < s$ and $\mathbf{c} > \mathbf{0}$. Then the following assertions hold:*

(a) $\max\limits_{i \in \Gamma_+} \dfrac{\Delta x_i}{x_i} = \dfrac{\Delta x_j}{x_j} \leqslant \dfrac{\Delta c_j}{c_j}$;

(b) $\min\limits_{i \in \Gamma_-} \dfrac{\Delta x_i}{x_i} = \dfrac{\Delta x_j}{x_j} \geqslant \dfrac{\Delta c_j}{c_j}$.

Proof. We only give the proof of (a). Let $\max_{i \in \Gamma_+} \Delta x_i / x_i = \Delta x_j / x_j$ for some $j \in \langle n \rangle$. Then Theorem 21 implies that $\Delta x_k / x_k \leqslant \Delta x_j / x_j$ for each $k \in \langle n \rangle$. Hence, $\lambda_k \geqslant \lambda_j$, or $(\lambda_k / \lambda_j) - 1 \geqslant 0$, for each $k \in \langle n \rangle$. Taking $\alpha = 1$ in (23), we find that

$$\sum_{k=1}^{n} t_{jk} \left(\frac{\lambda_k}{\lambda_j} - 1 \right) x_k' + \left(\frac{\mu_j}{\lambda_j} - 1 \right) c_j' = 0. \tag{33}$$

Hence, $(\mu_j / \lambda_j) - 1 \leqslant 0$, or $\mu_j \leqslant \lambda_j$, and so we find that $\Delta c_j / c_j \geqslant \Delta x_j / x_j$. ∎

THEOREM 25. *Let $T > 0$, $\mathbf{c} > \mathbf{0}$, $r < s$, and $\Gamma(\Delta\mathbf{c}) \neq \varnothing$. Then for each $i \notin \Gamma(\Delta\mathbf{c})$ the following holds:*

$$\min\left\{0, \inf_{j \in \Gamma_-} \frac{\Delta x_j}{x_j}\right\} < \frac{\Delta x_i}{x_i} < \max\left\{0, \sup_{j \in \Gamma_+} \frac{\Delta x_j}{x_j}\right\}. \tag{34}$$

Proof. Only the right-hand part of (34) will be proved. We may assume that $\Gamma_+ \neq \varnothing$ and that $\Delta x_j / x_j \geqslant 0$ for some $j \in \Gamma_+$ (see the proof of Theorem 21). Suppose to the contrary that there is an element $m \notin \Gamma$ such that $\Delta x_m / x_m = \max_{j \in \Gamma_+} \Delta x_j / x_j \geqslant 0$. Then Theorem 21 implies that $\Delta x_m / x_m = \max_{i \in \langle n \rangle} \Delta x_i / x_i \geqslant 0$. Hence, $\lambda_m = \min_{i \in \langle n \rangle} \lambda_i \leqslant 1$ and—as $m \notin \Gamma$—it follows that $\Delta c_m = 0$ or that $\mu_m = 1$. Taking $\alpha = 1$ in (23), we find that

$$\sum_{k=1}^{n} t_{mk} \left(\frac{\lambda_k}{\lambda_m} - 1 \right) x_k' + \left(\frac{1}{\lambda_m} - 1 \right) c_m = 0.$$

If $(1/\lambda_m)-1=0$, then $\Delta x_m/x_m=0$, and so it follows that $\max_{i\in\langle n\rangle}\Delta x_i/x_i=0$. This implies that $\max_{i\in\langle n\rangle}\Delta x_i=0$, and we find directly—see the proof of Theorem 21—that $\Gamma_+=\varnothing$, which contradicts the fact that $\Gamma_+=\varnothing$. Hence, $(1/\lambda_m)-1>0$. As $c_m>0$, we find that

$$\sum_{k=1}^{n} t_{mk}\left(\frac{\lambda_k}{\lambda_m}-1\right)x_k' < 0,$$

and—as $T>0$—it follows that $(\lambda_k/\lambda_m)-1<0$, or that $\lambda_k<\lambda_m$ for some $k\in\langle n\rangle$. This contradicts the fact that $\lambda_m\leqslant\lambda_i$ for each $i\in\langle n\rangle$. The conclusion is that in fact $\Delta x_i/x_i<\max_{j\in\Gamma_+}\Delta x_j/x_j$ for each $i\notin\Gamma$. ∎

This theorem asserts, in terms of the open Leontief model, that for any positive matrix with Perron-Frobenius eigenvalue less than one, the relative changes of the production in industries with unchanged demands are strictly between the maximal relative increase and minimal relative decrease of the productions of those industries in which the demands increase and decrease, respectively.

Corollary 26. *Let $T>0$, $\mathbf{c}>\mathbf{0}$, and $r<s$. Then the following assertions hold:*

(a) $\displaystyle\max_{i\in\Gamma_+}\frac{\Delta x_i}{x_i}=\frac{\Delta x_j}{x_j}<\frac{\Delta c_j}{c_j};$

(b) $\displaystyle\min_{i\in\Gamma_-}\frac{\Delta x_i}{x_i}=\frac{\Delta x_j}{x_j}>\frac{\Delta c_j}{c_j}.$

Proof. This corollary follows from Theorem 25 in the same way as Corollary 24 follows from Theorem 21. ∎

The interpretation of the assertions of Corollary 24(a) and 26(a) in terms of the open Leontief model is as follows: If the maximal relative increase of the production occurs in the jth industry, then the relative increase of the demand in this industry is greater than (or equal to) it.

13. MAXIMAL AND MINIMAL RELATIVE CHANGE OF DEMAND AND PRODUCTION

Lemma 27. *Let $r<s$. Then for each $i,j\in\langle n\rangle$ the following holds:*

$$x_i\Delta x_j - x_j\Delta x_i = \frac{1}{|sI-T|^2}\sum_{k<l}(c_{ki}c_{lj}-c_{kj}c_{li})(c_k\Delta c_l-c_l\Delta c_k).$$

Proof. Take any $i,j\in\langle n\rangle$. Then (15) implies that

$$
\begin{aligned}
x_i\Delta x_j - x_j\Delta x_i &= \left[\frac{1}{|sI-T|}\sum_{k=1}^{n} c_{ki}c_k\right]\left[\frac{1}{|sI-T|}\sum_{l=1}^{n} c_{lj}\Delta c_l\right] \\
&\quad -\left[\frac{1}{|sI-T|}\sum_{k=1}^{n} c_{kj}c_k\right]\left[\frac{1}{|sI-T|}\sum_{l=1}^{n} c_{li}\Delta c_l\right] \\
&= \frac{1}{|sI-T|^2}\sum_{k=1}^{n}\sum_{l=1}^{n}(c_{ki}c_{lj}-c_{kj}c_{li})c_k\Delta c_1 \\
&\qquad\qquad [\text{note that } c_{ki}c_{lj}-c_{kj}c_{li} = -(c_{li}c_{kj}-c_{lj}c_{ki})] \\
&= \frac{1}{|sI-T|^2}\sum_{k<l}(c_{ki}c_{lj}-c_{kj}c_{li})(c_k\Delta c_l - c_l\Delta c_k). \qquad ■
\end{aligned}
$$

Theorem 28. *Let $r<s$. Then the following holds:*

$$\frac{\Delta x_i}{x_i}=\frac{\Delta x_j}{x_j}\quad\text{for each } i,j\in\langle n\rangle \quad\Leftrightarrow\quad \frac{\Delta c_i}{c_i}=\frac{\Delta c_j}{c_j}\quad\text{for each } i,j\in\langle n\rangle.$$

Proof. It follows directly from Lemma 27 that if $\Delta c_i/c_i=\Delta c_j/c_j$ for each $i,j\in\langle n\rangle$, then $\Delta x_i/x_i=\Delta x_j/x_j$ for each $i,j\in\langle n\rangle$. In order to prove the reverse of this implication we write $B=sI-T$. Hence $\mathbf{c}=B\mathbf{x}$ and $\Delta\mathbf{c}=B\Delta\mathbf{x}$. It then follows that for each $i,j\in\langle n\rangle$

$$c_i\Delta c_j - c_j\Delta c_i = \sum_{k<h}(b_{ik}b_{jh}-b_{jk}b_{ih})(x_k\Delta x_h - x_h\Delta x_k),$$

quite similarly to the proof of Lemma 27.

Hence, if $\Delta x_i/x_i=\Delta x_j/x_j$ for each $i,j\in\langle n\rangle$, then also $\Delta c_i/c_i=\Delta c_j/c_j$ for each $i,j\in\langle n\rangle$. ■

This theorem asserts in terms of the open Leontief model that all relative changes of the demand are equal if and only if all relative changes of the production are equal.

Theorem 29. *Let $r<s$ and $n\geqslant 3$. Then the following assertions hold:*

(a) $c_{ii}c_{kj}-c_{ij}c_{ki}\geqslant 0$ *for each* $i,j,k\in\langle n\rangle$;

(b) *For each* $i,j\in\langle n\rangle$ *with* $i\neq j$ *the following holds: If* $c_{ii}c_{jj}-c_{ij}c_{ji}>0$, *then for each* $k \in \langle n \rangle$ *with* $i \neq k \neq j$ *it follows that*

$$c_{ii}c_{kj} - c_{ij}c_{ki} > 0 \quad \text{or} \quad c_{jj}c_{ki} - c_{ji}c_{kj} > 0.$$

Proof.

(a) Take any $i,j \in \langle n \rangle$, and choose $\mathbf{c}$ and $\Delta\mathbf{c}$ such that $\mathbf{c} \gg \mathbf{0}$ and $\Gamma(\Delta\mathbf{c}) = \Gamma_+(\Delta\mathbf{c}) = \{j\}$. Then Theorem 21 implies that $\Delta x_i/x_i \leqslant \Delta x_j/x_j$, and it follows from Lemma 26 that

$$0 \leqslant \frac{1}{|sI-T|^2} \sum_{k=1}^{n} (c_{ki}c_{jj} - c_{kj}c_{ji})c_k \Delta c_j.$$

As $|sI-T|^2 > 0$ and $\Delta c_j > 0$, we find that $\Sigma_{k=1}^n (c_{ki}c_{jj} - c_{kj}c_{ji})c_k \geqslant 0$. Hence, $c_{jj}c_{ki} - c_{ji}c_{kj} \geqslant 0$ for each $i,j,k \in \langle n \rangle$.

(b) Suppose to the contrary that there are indices $i,j,k \in \langle n \rangle$ with $i \neq j \neq k \neq i$ such that $c_{ii}c_{jj} - c_{ij}c_{ji} > 0, c_{ii}c_{kj} - c_{ij}c_{ki} = 0$, and $c_{jj}c_{ki} - c_{ji}c_{kj} = 0$. Then

$$c_{ki}c_{jj} = c_{kj}c_{ji},$$

$$c_{ki}c_{ij} = c_{kj}c_{ii}.$$

Multiplying the first equation by c_{ij} and the other one by c_{jj}, we find that $c_{ij}c_{kj}c_{ji} = c_{kj}c_{ii}c_{jj}$, or that $c_{kj}(c_{ii}c_{jj} - c_{ij}c_{ji}) = 0$, so that $c_{kj} = 0$. But this contradicts Theorem 11. Consequently, we find that $c_{ii}c_{kj} - c_{ij}c_{ki} > 0$ or $c_{jj}c_{ki} - c_{ji}c_{kj} > 0$. ∎

THEOREM 30. *Let* $r < s$. *There are vectors* $\mathbf{c} \gg \mathbf{0}$ *and* $\Delta\mathbf{c}$ *with corresponding vectors* $\mathbf{x}$ *and* $\Delta\mathbf{x}$ *such that for some* $i,j \in \langle n \rangle$ *the following holds*:

$$\frac{\Delta c_i}{c_i} \leqslant \frac{\Delta c_j}{c_j} \quad \text{and} \quad \frac{\Delta x_i}{x_i} > \frac{\Delta x_j}{x_j}.$$

Proof. $r < s$ implies that $|\mathrm{adj}(sI-T)| > 0$. So there is a $(2,2)$ submatrix of $\mathrm{adj}(sI-T)$ whose determinant is nonzero and hence—according to Theorem 29(a)—positive. Suppose that $c_{\alpha i}c_{\beta j} - c_{\beta i}c_{\alpha j} > 0$ for some $i,j,\alpha,\beta \in \langle n \rangle$ with $i \neq j$ and $\alpha \neq \beta$. Without loss of generality we may assume that $i < j$ and that $\alpha < \beta$. Lemma 27 now implies that for each $\mathbf{c}$, $\Delta\mathbf{c}$, $\mathbf{x}$, and $\Delta\mathbf{x}$,

$$x_i\Delta x_j - x_j\Delta x_i = \frac{1}{|sI-T|^2}\sum_{k<l}(c_{ki}c_{lj}-c_{kj}c_{li})(c_k\Delta c_l - c_l\Delta c_k)$$

$$= \frac{1}{|sI-T|^2}\Bigg[(c_{ii}c_{jj}-c_{ij}c_{ji})(c_i\Delta c_j - c_j\Delta c_i)$$

$$+(c_{\alpha i}c_{\beta j}-c_{\beta i}c_{\alpha j})(c_\alpha\Delta c_\beta - c_\beta\Delta c_\alpha)$$

$$+\sum_{\substack{k<l\\(i,j)\neq(k,l)\neq(\alpha,\beta)}}(c_{ki}c_{lj}-c_{kj}c_{li})(c_k\Delta c_l - c_l\Delta c_k)\Bigg].$$

We distinguish two cases:

(1) $i\neq\alpha$ *or* $j\neq\beta$. Suppose that $i\neq\alpha$. Clearly, $\mathbf{c}$ and $\Delta\mathbf{c}$ may be chosen such that $c_\alpha\Delta c_\beta - c_\beta\Delta c_\alpha<0$, $c_i\Delta c_j - c_j\Delta c_i\geqslant 0$, and $c_\alpha\Delta c_\beta - c_\beta\Delta c_\alpha$ is sufficiently small that $x_i\Delta x_j - x_j\Delta x_i<0$. Hence $\Delta c_i/c_i\leqslant\Delta c_j/c_j$ and $\Delta x_i/x_i>\Delta x_j/x_j$.

(2) $i=\alpha$ *and* $j=\beta$. Hence $c_{ii}c_{jj}-c_{ij}c_{ji}>0$. It follows from Theorem 28(b) that $[c_{ii}c_{\sigma j}-c_{ij}c_{\sigma i}>0$ or $c_{jj}c_{\sigma i}-c_{ji}c_{\sigma j}>0]$ for each $\sigma\in\langle n\rangle$ with $i\neq\sigma\neq j\neq i$. Suppose that $c_{ii}c_{\sigma j}-c_{ij}c_{\sigma i}>0$ and that $i<\sigma$. It follows then, in the same way as in (1) (by taking $\alpha=i$ and $\beta=\sigma$), that $\mathbf{c}$ and $\Delta\mathbf{c}$ may chosen such that $\Delta c_i/c_i\leqslant\Delta c_j/c_j$ and $\Delta x_i/x_i>\Delta x_j/x_j$. ■

In terms of the open Leontief model this theorem means that, in general, the maximal or minimal relative changes of the demand and the production do not occur in the same industry. In fact, it is always possible to find a demand vector $\mathbf{c}$ and a demand change vector $\Delta\mathbf{c}$ such that $\Delta c_j/c_j$ is maximal or minimal, but $\Delta x_j/x_j$ is not maximal or minimal, respectively.

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